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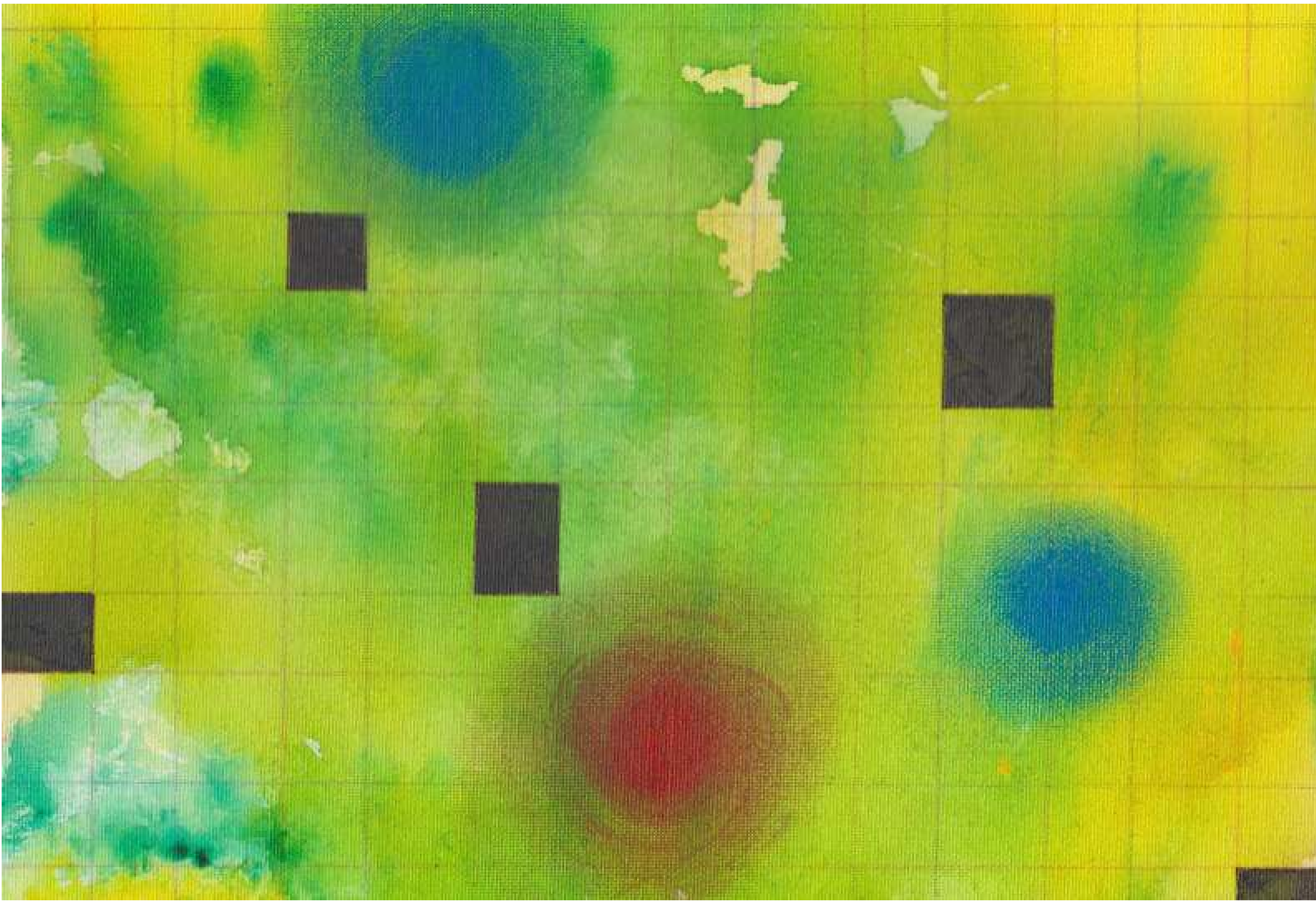
GENERALISED IMPULSE RESPONSE FUNCTION AS A PERTURBATION OF A GLOBAL SOLUTION TO DSGE MODELS



CONTENTS

ABBREVIATIONS

DSGE – dynamic stochastic general equilibrium models
GIRF – generalised impulse response function
i.i.d. – independent and identically distributed
IRF – impulse response function

ABSTRACT

In the conventional perturbation approach for solving DSGE models, the dynamics of the deviation of solutions from the steady state after a shock hitting an economy represents an impulse response function (IRF). A method to construct the IRF as a deviation from a deterministic global solution is proposed. The approach detects asymmetric reactions of an economy to shocks in different initial conditions. For example, in an economic downturn a negative shock might affect the economy more severely than in normal economic conditions. The method allows for constructing the IRF for highly nonlinear DSGE models.

Keywords: DSGE, perturbation, global solution, trend inflation

JEL codes: C62, D58, D84

The views expressed in this paper are those of the author – an employee of the Monetary Policy Department of Latvijas Banka, and do not necessarily reflect the stance of Latvijas Banka. The author assumes sole responsibility for any errors and omissions. E-mail address: Viktors.Ajevskis@bank.lv.

1. INTRODUCTION

Perturbation methods are the most widely-used approach to solve DSGE models owing to their ability to deal with medium and large size models – in a small computational time. The approach is based on the Taylor series approximation of DSGE models around the steady state and has recently been extensively developed (see, for example, Collard and Juillard (2001), Gaspar and Judd (1997), Gomme and Klein (2011), Jin and Judd (2002), Judd and Guu (1997), Lombardo (2010), Lombardo and Uhlig (2014), and Schmitt-Grohé and Uribe (2004)). However, the solutions obtained by this type of methods are accurate only in some small neighbourhood of the steady state. Out of the neighbourhood, the solutions may behave odd, for example, they can imply explosive dynamics or be very inaccurate (Kim et al. (2008) and Den Haan and De Wind (2012)). Moreover, depending on whether an economy is in the steady state or far away from that, it may react differently to a shock. This situation may be the case if the economy is: (i) highly indebted and/or leveraged, (ii) in a deep recession, (iii) a transition and developing one, or (iv) at the effective lower bound for nominal interest rate or other binding constraints.

Global methods are used to deal with solving DSGE models in global domains. These methods are quite different, depending on the types of models solved – stochastic or deterministic ones. Projection methods are used for solving stochastic models, however, they suffer from the curse of dimensionality and as such they can deal with only small-scale models (Judd (1998), Aruoba et al. (2006) and Heer and Maussner (2008)). On the other hand, the approaches based on Newton's method (see, for example, a bunch of algorithms available in Dynare (Adjemian et al. (2011)) are used to solve deterministic models. In this case, global solutions can be obtained reasonably fast even for large size models.

This study presents an approach that combines both stochastic perturbation and deterministic Newton's methods. Specifically, the approach implies the linearization of a DSGE model along a known global deterministic solution. In the conventional perturbation approach, after linearization around the steady state, one needs to solve a rational expectations model with constant coefficients (see Blanchard and Kahn (1980), Anderson and Moore (1985), Uhlig (1999), Klein (2000) and Sims (2002)). While, the linearization around a deterministic solution gives us a rational expectations model with deterministic time-varying coefficients. The current paper uses backward induction to solve this type of models. As a result, we obtain a time-varying coefficient moving-average representation of the solution. As in the standard case, the coefficients of this representation may be treated as an IRF. The approach presented is based on the idea of perturbation around a deterministic path by Ajevskis (2017). However, while in Ajevskis (2017) the main focus is on the construction of policy functions, this paper investigates the generalised impulse response functions (GRIFs) under global initial conditions.

We apply the proposed method to the model of trend inflation by Ascari and Sbordone (2014). They find that the value of the steady state inflation rate affects significantly the economic dynamics. However, Ascari and Sbordone (2014) consider the response of the economy to shocks when it stays at the steady state. By contrast, we address the issue of reaction to a monetary shock when the economy is away from the steady state. Specifically, the initial conditions correspond to the zero nominal interest rate, $i = 0$. These conditions distinguish reasonably from the steady state. On the other hand, Ascari and Sbordone (2014) show that the New Keynesian model is especially nonlinear around the zero inflation steady state. This observation allows for using our global approach not moving too far from the steady state.

The obtained results show that for a non-zero inflation steady state the initial reaction of inflation to 0.1% negative interest rate shock is significantly smaller than the reaction obtained for the model linearized around the steady state and for the third order Taylor series approximation with initial value at $i = 0$. For the 4% steady state of the inflation rate, the IRF even changes its pattern becoming hump-shaped.

This paper also relates to (Andreasen et al. (2013)) who construct generalised IRFs (GIRFs) conditional on the state of the economy. For doing this, they use the pruning of the third order Taylor series approximation of the DSGE model and then perform Monte Carlo simulation for the path of variables under model dynamics. However, as mentioned above, the Taylor series approximation might be very poor if a solution is far away from the steady state. Besides, the use of Monte Carlo simulation is rather time-consuming.

2. THE MODEL

This section describes the DSGE models under consideration in general form as well as notations, preparatory material and derivations needed in Section 3. We consider DSGE models in the form:

$$E_t f(y_{t+1}, y_t, y_{t-1}, u_t) = 0 \qquad (1)$$

where E_t denotes the conditional expectations operator, y_t is an $n_y \times 1$ vector containing the t-period endogenous variables; $u_t = \sigma\varepsilon_t$ is a vector of exogenous i.i.d. stochastic shocks; σ ($\sigma > 0$) is a scaling parameter; ε_t is an auxiliary random variable with the covariance matrix Ω. The mapping f maps $\mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_y}$ into $\mathbb{R}^{n_y}$ and is assumed to be sufficiently smooth.

If $\sigma = 0$ all shocks disappear and the system (1) becomes deterministic

$$E_t f(y_{t+1}, y_t, y_{t-1}, 0) = 0 \qquad (2).$$

The solution, $y_t^{(0)}$, to the deterministic problem must satisfy the initial conditions

$$y_0^{(0)} = y_0 \qquad (3)$$

and the terminal ones

$$y_\infty^{(0)} = \bar{y} \qquad (4).$$

The terminal conditions for y_t are supposed to be at the deterministic steady state, i.e. vectors defined by the equation $f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$. The deterministic problems (2)–(4) can be solved globally for a reasonable computational time by a number of effective algorithms, for example, the extended path method (Fair and Taylor (1983)) or a Newton-like method (for example, Juillard (1996)).

If we linearize the model around the obtained deterministic solution, (1) takes the form:

$$E_t\{f_{1,t}\hat{y}_{t+1} + f_{2,t}\hat{y}_t + f_{3,t}\hat{y}_{t-1} + \sigma f_{4,t}\varepsilon_t\} = 0 \qquad (5)$$

where $\hat{y}_t = y_t - y_t^{(0)}$ is deviation from the deterministic solution. The matrices

$$f_{i,t} = f_i\left(y_{t+1}^{(0)}, y_t^{(0)}, y_{t-1}^{(0)}, 0\right), i = 1, \dots, 4 \tag{6}$$

are the Jacobian matrices of the mapping f with respect to the ith argument (i.e. y_{t+1}, y_t, y_{t-1} and u_t, respectively), at the points of the deterministic solution $(y_{t+1}^{(0)}, y_t^{(0)}, y_{t-1}^{(0)}, 0)$. Equation (5) can also be obtained by expanding original equation (1) in a small parameter σ, as it is done in applied mathematics literature (see, for example, Holmes (2013) and Nayfeh (1973)). Now we have a time-varying rational expectation model as $f_{i,t}$ depend on t.

We can represent $f_{i,t}$ in (6) as $f_{i,t} = \bar{f}_i + \hat{f}_{i,t}$, $i = 1, \dots, 4$ where

$$\bar{f}_i = f_i(\bar{y}, \bar{y}, \bar{y}, 0)$$

are the Jacobian matrices of the mapping f at the steady state with respect to the ith argument, and

$$\hat{f}_{i,t} = f_{i,t}(y_{t+1}^{(0)}, y_t^{(0)}, y_{t-1}^{(0)}, 0) - f_i(\bar{y}, \bar{y}, \bar{y}, 0) \tag{7}.$$

Note also that $\hat{f}_{i,t} \to 0$ as $t \to \infty$, because a deterministic solution must tend to the deterministic steady state as t tends to infinity. Consequently, $f_{i,t}$ can be thought of as a time-varying perturbation of f_i.

Relabeling $\hat{y}_{t-1}$ as $\widehat{w}_t$, we can rewrite (5) as a system of equations

$$\begin{aligned} \widehat{w}_{t+1} &= \hat{y}_t \\ (f_1 + \hat{f}_{1,t})E_t\{\hat{y}_{t+1}\} &= -(f_2 + \hat{f}_{2,t})\hat{y}_t - (f_3 + \hat{f}_{3,t})\widehat{w}_t \end{aligned} \tag{8}.$$

Here we take into account that $E_t \varepsilon_{t+1} = 0$.

Equation (8) can be written in the vector form

$$\Phi_t E_t \begin{bmatrix} \widehat{w}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix} = \Lambda_t \begin{bmatrix} \widehat{w}_t \\ \hat{y}_t \end{bmatrix} \tag{9}$$

where

$$\Phi_t = \begin{bmatrix} I & 0 \\ 0 & f_1 + \hat{f}_{1,t} \end{bmatrix}$$

and

$$\Lambda_t = \begin{bmatrix} 0 & I \\ f_3 + \hat{f}_{3,t} & f_2 + \hat{f}_{2,t} \end{bmatrix}.$$

Here I is the identity matrix. Pre-multiplying (9) by Φ_t^{-1}, we get

$$E_t \begin{bmatrix} \widehat{w}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix} = L \begin{bmatrix} \widehat{w}_t \\ \hat{y}_t \end{bmatrix} + M_t \begin{bmatrix} \widehat{w}_t \\ \hat{y}_t \end{bmatrix} \tag{10}.$$

Notice that $\lim_{t\to\infty} M_t = 0$. As in the case of rational expectations models with constant parameters, it is convenient to transform (10) using the spectral property of L. Namely, the matrix L is transformed into a block-diagonal one

$$L = ZPZ^{-1} \tag{11}$$

where

$$P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{12}$$

where A and B are matrices with eigenvalues larger and smaller than one (in modulus), respectively, and Z is an invertible matrix. This can be done, for example, by initially transforming L into a simple Schur form $L = Z_1 L_1 Z_1^{-1}$ where Z_1 is a unitary matrix, L_1 is an upper triangular Schur form with the eigenvalues along the diagonal (see Golub and Van Loan (1996)). We then transform the matrix L_1 into the block-diagonal Schur factorization $L_1 = Z_2 P Z_2^{-1}$ where Z_2 is an invertible matrix and P is block diagonal, and each diagonal block is a quasi upper-triangular Schur matrix. Hence, the matrix Z in (11) has the form $Z = Z_1 Z_2$. We also impose the conventional Blanchard–Kahn condition (Blanchard and Kahn (1980)) on the dimension of unstable subspace, i.e. it is equal to the number of forward-looking variables.

After introducing the auxiliary variables

$$[s_t, u_t]' = Z^{-1}[\widehat{w}_t, \hat{y}_t]' \tag{13}$$

and pre-multiplying (10) by Z^{-1}, we have

$$E_t s_{t+1} = A s_t + Q_{11,t} s_t + Q_{12,t} u_t, \tag{14}$$

$$E_t u_{t+1} = B u_t + Q_{21,t} s_t + Q_{22,t} u_t, \tag{15}$$

where

$$\begin{bmatrix} Q_{11,t} & Q_{12,t} \\ Q_{21,t} & Q_{22,t} \end{bmatrix} = Z^{-1} M_t Z \qquad (16).$$

System (14)–(15) is a linear rational expectations model with time-varying parameters. The matrix Z in (11) can be chosen in such a way that

$$\parallel A \parallel < \alpha + \gamma < 1 \text{ and } \parallel B^{-1} \parallel < \beta + \gamma < 1 \tag{17}$$

where α and β are the largest eigenvalues (in modulus) of the matrices A and B^{-1}, respectively, and γ is arbitrarily small (see Hartmann (1982)). Note also that $\parallel B \parallel^{-1} < 1$ for sufficiently small γ.

Let

$$B_t = B + Q_{22,t}, A_t = A + Q_{11,t} \qquad (18),$$

then (14)–(15) can be rewritten in the form

$$E_t s_{t+1} = A_t s_t + Q_{12,t} u_t \qquad (19),$$

$$E_t u_{t+1} = B_t u_t + Q_{21,t} s_t \qquad (20).$$

Assume that $T + 1 > 0$ is a terminal moment that guarantees the necessary accuracy of the solution, $(x_t^{(0)}, y_t^{(0)})$, $t = 0, \dots, T + 1$, to the deterministic problem (2).

3. SOLVING THE RATIONAL EXPECTATIONS MODEL WITH TIME-VARYING PARAMETERS

From (20) for the time T we have

$$u_T = -B_T^{-1}Q_{21,T}s_T + B_T^{-1}E_T u_{T+1}.$$

Assume now that by the time $T+1$ the solution to the linear rational expectations model with time-varying parameters (19)–(20) converges to zero, i.e. $E_T u_{T+1} = 0$. Denoting $K_{T,T} = B_T^{-1}Q_{21,T}$ gives

$$u_T = -K_{T,T}s_T \tag{21}$$

For $T-1$ we have

$$u_{T-1} = -B_{T-1}^{-1}Q_{21,T-1}s_{T-1} + B_{T-1}^{-1}E_{T-1}u_T \tag{22}$$

Taking conditional expectations at the time $T-1$ from both sides (21) and inserting $E_{T-1}s_T$ from (19), we get

$$E_{T-1}u_T = -K_{T,T}(A_{T-1}s_{T-1} + Q_{12,T-1}u_{T-1}) \tag{23}$$

Inserting (23) into (22) gives

$$u_{T-1} = -B_{T-1}^{-1}Q_{21,T-1}s_{T-1} + B_{T-1}^{-1}[-K_{T,T}(A_{T-1}s_{T-1} + Q_{12,T-1}u_{T-1})].$$

Reshuffling terms, we have

$$(I + B_{T-1}^{-1}K_{T,T}Q_{12,T-1})u_{T-1} = -B_{T-1}^{-1}(Q_{21,T-1} + K_{T,T}A_{T-1})s_{T-1}. \tag{24}$$

Multiplying (24) by $(I + B_{T-1}^{-1}K_{T,T}Q_{12,T-1})^{-1}$ yields

$$u_{T-1} = -(I + B_{T-1}^{-1}K_{T,T}Q_{12,T-1})^{-1}B_{T-1}^{-1}(Q_{21,T-1} + K_{T,T}A_{T-1})s_{T-1}.$$

Denoting $L_{T,T-1} = (B_{T-1} + K_{T,T}Q_{12,T-1})$, we obtain

$$u_{T-1} = -L_{T,T-1}^{-1}(Q_{21,T-1} + B_{T-1}^{-1}K_{T,T}A_{T-1})s_{T-1}.$$

Denoting

$$K_{T,T-1} = L_{T,T-1}^{-1}(Q_{21,T-1} + B_{T-1}^{-1}K_{T,T}A_{T-1})$$

and

$$R_{T-1} = L_{T,T-1}^{-1}(\Pi_{2,T-1} + K_{T,T}\Pi_{1,T-1} + R_T\Lambda),$$

we have

$$u_{T-1} = -K_{T,T-1}s_{T-1}.$$

Proceeding further with backward recursion, we obtain finite-horizon solutions for each $t = 0,1,2,\ldots,T+1$

$$u_t = -K_{T,t}s_t \tag{25}$$

where $K_{T,t}$ can be computed by backward recursion

$$K_{T,t} = L_{T,t}^{-1}(Q_{21,t} + B_t^{-1}K_{T,t+1}A_t).$$

Inserting (25) into (14) gives

$E_t s_{t+1} = A s_t + (Q_{11,t+1} - Q_{12,t+1} K_{T,t}) s_t.$

Denoting $\mathbb{A}_t = A_{t+1} + Q_{11,t+1} - Q_{12,t+1} K_{T,t}$, we have

$E_t s_{t+1} = \mathbb{A}_t s_t$ (26).

It is easy to see that

$$\begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} - E_t \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbb{R}_{1,t} \\ \mathbb{R}_{2,t} \end{bmatrix} \varepsilon_{t+1} = Z \Phi_t^{-1} f_{4,t} \sigma \varepsilon_{t+1}.$$

From (26) it follows that

$(E_t s_{t+1} - s_{t+1}) + s_{t+1} = \mathbb{A}_t s_t,$

thus we obtain

$s_{t+1} = \mathbb{A}_t s_t + \mathbb{R}_{1,t} \varepsilon_{t+1}$ (27).

Assuming that $s_0 = 0$, for $t = 1$ from (27) we have

$s_1 = \mathbb{R}_{1,0} \varepsilon_1;$

for $t = 2$

$s_2 = \mathbb{A}_1 \mathbb{R}_{1,0} \varepsilon_1 + \mathbb{R}_{1,1} \varepsilon_2.$

Continuing in this fashion, we get the moving-average representation of s_t:

$s_t = \gamma_{t,t} \varepsilon_t + \gamma_{t,t-1} \varepsilon_{t-1} + \cdots + \gamma_{t,2} \varepsilon_2 + \gamma_{t,1} \varepsilon_1,$ (28)

where the coefficients $\gamma_{t,t-i}$ can be obtained by forward recursion

$\gamma_{t,t} = \mathbb{R}_{1,t-1},$

$\gamma_{t,t-1} = \mathbb{A}_{t-1} \gamma_{t-1,t-1}$

…

$\gamma_{t,t-i} = \mathbb{A}_{t-1} \gamma_{t-1,t-i}$

…

$\gamma_{t,1} = \mathbb{A}_{t-1} \gamma_{t-1,1}.$

For the variable u_t we also have a moving-average representation. Inserting (28) in (25), we have

$u_t = -K_{T,t} (\gamma_{t,t} \varepsilon_t + \cdots + \gamma_{t,1} \varepsilon_1)$ (29)

or in the shorter form

$u_t = \delta_{t,t} \varepsilon_t + \delta_{t,t-1} \varepsilon_{t-1} + \cdots + \delta_{t,2} \varepsilon_2 + \delta_{t,1} \varepsilon_1,$ (30)

where $\delta_{t,i} = -K_{T,t} \gamma_{t,i}$. Taking into account that $\hat{x}_t = Z_{11} s_t + Z_{12} u_t$ and $\hat{y}_t = Z_{21} s_t + Z_{22} u_t$, we get the moving-average representation for original variables

$$\hat{x}_t = \rho^x_{t,t}\varepsilon_t + \rho^x_{t,t-1}\varepsilon_{t-1} + \cdots + \rho^x_{t,2}\varepsilon_2 + \rho^x_{t,1}\varepsilon_1,$$

$$\hat{y}_t = \rho^y_{t,t}\varepsilon_t + \rho^y_{t,t-1}\varepsilon_{t-1} + \cdots + \rho^y_{t,2}\varepsilon_2 + \rho^y_{t,1}\varepsilon_1,$$

where $\rho^x_{t,i} = Z_{11}\gamma_{t,i} + Z_{12}\delta_{t,i}$ and $\rho^y_{t,i} = Z_{21}\gamma_{t,i} + Z_{22}\delta_{t,i}$.

4. EXAMPLE

To illustrate how the presented method works, we apply it to the DSGE model of trend inflation (Ascari and Sbordone (2014)) where the authors show the importance of taking into account the non-zero inflation steady state for the dynamical properties of the model economy. In this model the representative agent maximizes the intertemporal utility function

$$\max_{(C_t,N_t)} E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} - d_n e^{\zeta} \frac{N_t^{1+\phi}}{1+\phi} \quad (31),$$

subject to

$$P_t C_t + (1+i_t)^{-1} B_t = W_t N_t - T_t + D_t + B_{t-1} \quad (32)$$

where c_t is consumption, N_t is the labour input, i_t is the nominal interest rate, B_t is one-period bond holdings, W_t is the nominal wage rate, T_t is lump sum taxes, D_t is the profit income, ζ_t is a labour supply shock, β is the discount factor, and E_0 is the conditional expectations operator.

Final goods producers use the following technology:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad (33).$$

Their demand for intermediate goods producers is

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_{t+j}}\right) Y_{t+j} \quad (34).$$

The production function of intermediate goods producers:

$$Y_{i,t} = A_t N_{i,t}^{1-\alpha} \quad (35).$$

A_t is an exogenous stationary process for the level of technology.The labour demand and the real marginal costs of firm i are:

$$N_{i,t}^d = \left(\frac{Y_{i,t}}{A_t}\right)^{\frac{1}{1-\alpha}} \quad (36),$$

$$MC_t = \frac{W_t}{P_t} \frac{1}{1-\alpha} A_t^{\frac{1}{\alpha-1}} Y_{i,t}^{\frac{\alpha}{1-\alpha}} \quad (37).$$

The model is described by the following equations:

$$\frac{1}{Y_t^{\sigma}} = \beta E_t \frac{(1+i_t)}{\pi_{t+1} Y_{t+1}^{\sigma}} \quad (38),$$

$$W_t = Y_t^{\sigma} d_n e^{\zeta_t} N_t^{\varphi} \quad (39),$$

$$p_t^* = \left(\frac{1-\theta\pi_{t-1}^{(1-\varepsilon)\rho}\pi_t^{\varepsilon-1}}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} \quad (40),$$

$$(p_t^*)^{1+\frac{\varepsilon\,\alpha}{1-\alpha}} = \frac{\varepsilon}{(\varepsilon-1)\,(1-\alpha)}\,\frac{\psi_t}{\phi_t} \quad (41),$$

$$\psi_t = W_t\, A_t^{\frac{(-1)}{1-\alpha}}\, Y_t^{\frac{1}{1-\alpha}-\sigma} + \beta\,\theta\,\pi_t^{\frac{\varepsilon\,(-\rho)}{1-\alpha}}\,\pi_{t+1}^{\frac{\varepsilon}{1-\alpha}}\,\psi_{t+1} \quad (42),$$

$$\phi_t = Y_t^{1-\sigma} + \beta\,\theta\,\pi_t^{(1-\varepsilon)\,\rho}\,\pi_{t+1}^{\varepsilon-1}\,\phi_{t+1} \quad (43),$$

$$N_t = s_t\,\left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \quad (44),$$

$$s_t = (1-\theta)\,\left({p^*}_t\right)^{\frac{(-\varepsilon)}{1-\alpha}} + \theta\,(\pi_{t-1})^{\frac{\rho\,(-\varepsilon)}{1-\alpha}}\,(\pi_t)^{\frac{\varepsilon}{1-\alpha}}\,s_{t-1} \quad (45),$$

$$\frac{1+i_t}{1+\bar{\imath}} = \left(\frac{1+i_{t-1}}{1+\bar{\imath}}\right)^{\rho_i}\left(\left(\frac{\pi_t}{\bar{\pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y}\right)^{1-\rho_i} e^{\nu_t} \quad (46),$$

$$MC_t = w_t\,\frac{1}{1-\alpha}\,(A_t)^{\frac{1}{\alpha-1}}\,(y_t)^{\frac{\alpha}{1-\alpha}} \quad (47),$$

$$(r_t) = \frac{1+(i_t)}{(\pi_{t+1})} \quad (48),$$

$$U_t = y_t - \frac{d_n\,(\zeta_t)\,(N_t)^{1+\varphi}}{1+\varphi} + \beta\,U_{t+1} \quad (49),$$

$$\nu_t = \rho_\nu\,\nu_{t-1} + \varepsilon_{\nu t} \quad (50),$$

$$A_t = \rho_a\,A_{t-1} + \varepsilon_{A t} \quad (51),$$

$$\zeta_t = \rho_\zeta\,\zeta_{t-1} + \varepsilon_{\zeta t} \quad (52),$$

$$(\tilde{A}_t) = \frac{(A_t)}{(s_t)} \quad (53).$$

The calibrated parameters follow Ascari and Sbordone (2014): $\sigma = 1$, $\phi = 1$ and 3, $\varepsilon = 10$, $\phi_\pi = 2$, $\phi_Y = 0.5/4$, $\beta = 0.99$, $\theta = 0.75$ and $\rho_i = 0.8$. The calibrated parameters are given in Table 1.

Table 1
Calibrated parameters

σ	ϕ	ε	ϕ_π	ϕ_Y	β	θ	ρ_i
1	2	10	2	0.125	0.99	0.75	0.8

The deterministic solution is obtained for the economy with initial point at the nominal interest rate $i_0 = 0$ (all other state variables are at their steady state). This economic condition corresponds to the current situation in the advanced countries. We do not model the effective lower bound for the short-term nominal interest rate, assuming that this bound lies low enough below zero.

We consider the IRFs of the inflation rate to the negative 0.1% interest rate shock for three values of inflation steady state: 0%, 2% and 4%. Choosing the 4% steady state inflation is related to a recent debate on the proposal by some researchers (Blanchard et al. (2010) and Williams (2009) and Ball (2013)) to adopt a 4% inflation target in the United States of America in order to mitigate the zero bound constraint.[1] Note that it is the usual practice in DSGE modeling for monetary analysis to assign the steady state value of the inflation rate to zero. We compare the IRFs obtained from three methods: (a) linearization around the steady state, (b) the third order Taylor series approximation around the deterministic steady state, and (c) the state-depended global GIRFs. In the case of the third order Taylor series, approximation of the GIRF is constructed as a difference between shocked and unshocked solutions both having the same initial conditions corresponding to the nominal interest rate $i_0 = 0$. For this purpose, we use the Dynare command simult_ that allows for constructing Taylor series approximate solution starting from a given point and shocked by given innovations.

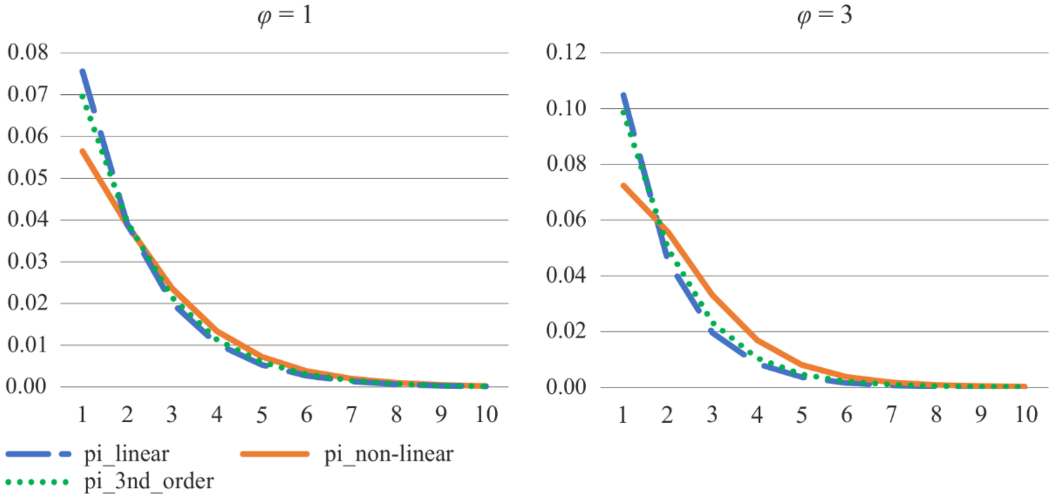
Figures 1–3 show the IRF of inflation rate to the 0.1% negative interest rate shock for various steady state inflation rates and the Frish elasticity, φ, equal to 1 (left panel) and 3 (right panel). For all steady states of the inflation rate, the reaction of inflation to the interest rate shock is considerably smaller for the case of linearization around the deterministic solution than for linearization and the third order Taylor approximation around the steady state. This difference increases with the steady state inflation rate rising. The relative differences between shocks of the linearized around the steady state model and that linearized around the deterministic solution at $t = 1$ for $\varphi = 1$ is 34%, 57% and 84% for the steady state inflation rates of 0%, 2% and 4%, respectively; whereas for $\varphi = 3$ these figures are 45%, 93% and 171%, respectively. The relative differences between shocks of the linearised around the deterministic solution and the third order approximation around the steady state at $t = 1$ for $\varphi = 1$ is 23%, 36% and 46% for the steady state inflation rates of 0%, 2% and 4%, respectively; whereas for $\varphi = 3$ these figures are 36%, 67% and 91%, respectively. The difference is quite large for the steady state inflation of 4% for both values of φ. On the other hand, the higher value of the Frish elasticity implies the larger relative difference between two shocks due to the higher degree of model nonlinearity.

The IRF becomes also more persistent with a higher Frish elasticity coefficient. For $\varphi = 3$ and the steady state inflation 4% in the case of model linearized around the deterministic solution even the pattern of IRF changes. It becomes hump-shaped, whereas for perturbations around the steady state it remains monotonically decreasing. The difference between the first order and third order approximations is quite small for the steady state inflation of 0% and 2%.

Ascari and Sbordone (2014) show that because of the flattening of the Phillips curve with higher steady state inflation, at the time $t = 1$, the sensitivity of inflation to an interest rate shock is reduced. It follows from our results that this observation is more pronounced if the initial conditions are not the steady state. Therefore, in the case of perturbation around the deterministic solution the inflation rate is less sensitive to an interest rate shock than in the case of perturbation around the steady state at the time $t = 1$, if an economy stays at the zero short-term interest rate even without consideration of the effect of the lower bound for the nominal interest rate.

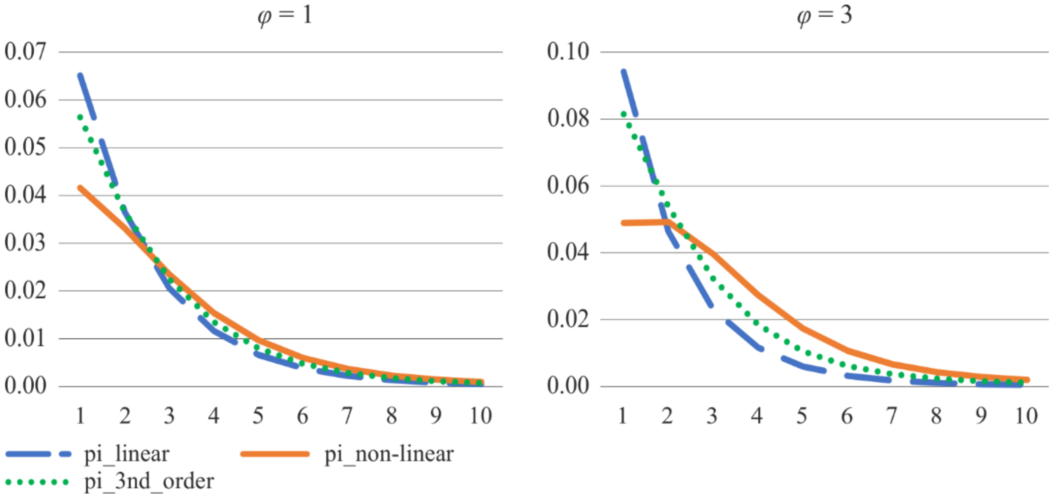
[1] The same values of inflation steady state are used in Ascari and Sbordone (2014).

Figure 1
Steady state inflation 0%



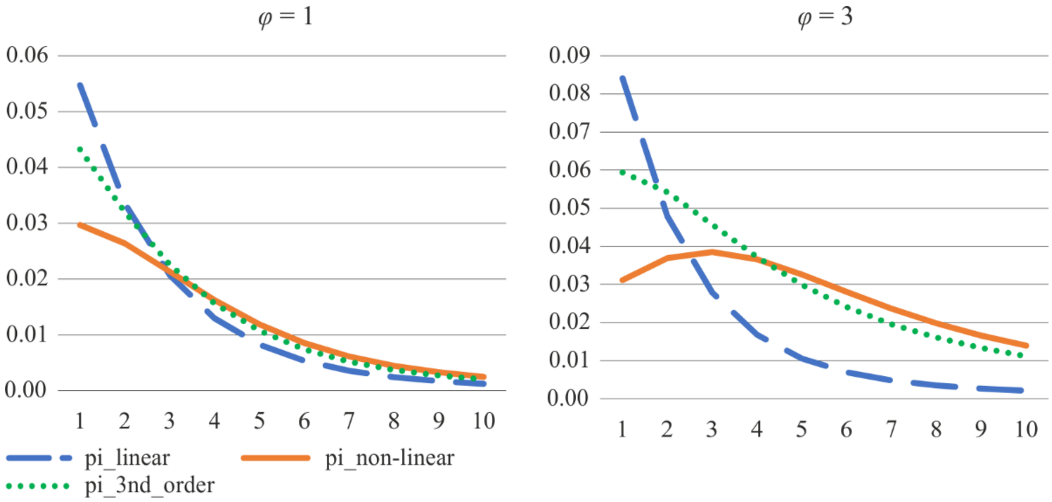
Notes. IRFs for inflation to the 0.1% negative interest rate shock for steady state inflation 0% and the Frish elasticities $\varphi = 1$ (left panel) and 3 (right panel). The dashed blue line is the IRFs for the model linearized around the steady state, the solid orange line is IRFs for the model linearized around the deterministic solution.

Figure 2
Steady state inflation 2%



Notes. IRFs for inflation to the 0.1% negative interest rate shock for steady state inflation 2% and the Frish elasticities $\varphi = 1$ (left panel) and 3 (right panel). The dashed blue line is the IRFs for the model linearized around the steady state, the solid orange line is IRFs for the model linearized around the deterministic solution.

Figure 3
Steady state inflation 4%



Notes. IRFs for inflation to the 0.1% negative interest rate shock for steady state inflation 4% and the Frish elasticities φ = 1 (left panel) and 3 (right panel). The dashed blue line is the IRFs for the model linearized around the steady state, the solid orange line is IRFs for the model linearized around the deterministic solution.

5. CONCLUSIONS

This study proposes an approach based on a perturbation around a deterministic path for constructing GIRF for DSGE models. The approach allows for dealing with highly nonlinear models and situations where initial conditions are far away from the steady state. Under the assumption that the deterministic solution to the model is already found, the method reduces the problem to solving recursively a linear rational expectations model with deterministic time-varying parameters. The approach is applied to the DSGE model with the inflation trend and zero initial value for interest rate. The results show that the difference in impulse response functions is significant if we take account of the initial value of the economy. The model framework is quite suitable for estimation by the Kalman filter with deterministic time-varying coefficients.

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